Tutorial 8

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We shall begin by briefly reviewing the notion of orientation for surfaces.

As we have seen in Sec. 2-4, given a parametrization $\mathbf{x}: U \subset R^2 \to S$ of a regular surface S at a point $p \in S$, we can choose a unit normal vector at each point of $\mathbf{x}(U)$ by the rule

$N(u,v)$.

Gauss map. $$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q), \qquad q \subset \mathbf{x}(U).$$

Thus, we have a differentiable map $N: \mathbf{x}(U) \to R^3$ that associates to each $q \in \mathbf{x}(U)$ a unit normal vector $N(q)$.

More generally, if $V \subset S$ is an open set in S and $N: V \to R^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q, we say that *N is a differentiable field of unit normal vectors on V*.

DEFINITION 1. *Let* S $\subset$ R³ *be a surface with an orientation* N. *The map* N: S $\to$ R³ *takes its values in the unit sphere*

$$S^2 = \{(x, y, z) \in R^3;$$

The map N: S $\to$ S², *thus defined, is called the* Gauss map *of* S (*Fig. 3-2*).†

It is straightforward to verify that the Gauss map is differentiable. The differential dN_p of N at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since $T_p(S)$ and $T_{N(p)}(S^2)$ are parallel planes, dN_p can be looked upon as a linear map on $T_p(S)$.

$dN_p : T_p(S) \longrightarrow T_p(S^2)$. Tangent map.

Consider curve $\alpha(t)$ in S. $\alpha(0) = p$.

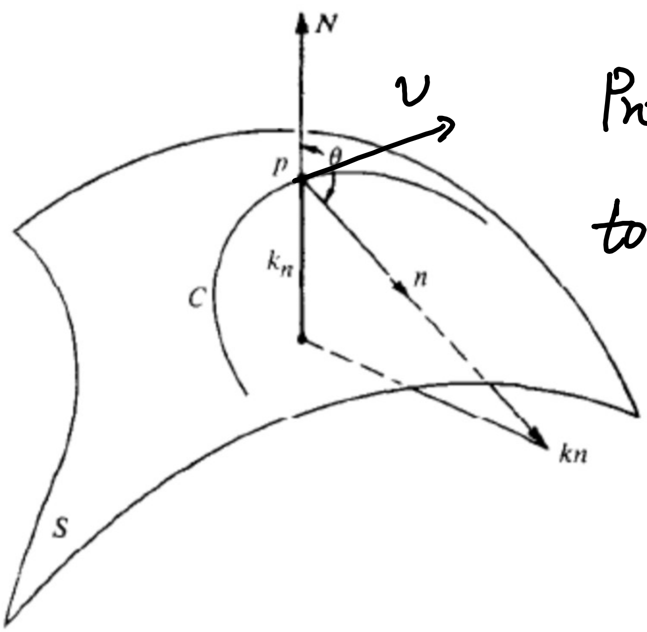
$$dN_p(\alpha'(0)) \triangleq \frac{d}{dt} N(u(t), v(t))\Big|_{t=0}.$$

$$= u'(0) N_u + v'(0) N_v.$$

$\mathrm{II}_p(v) \triangleq \langle -dN_p(v), v\rangle$: <u>self-adjoint</u> operator $-dN_p(v)$.

DEFINITION 3. *Let* C *be a regular curve in* S *passing through* p $\in$ S, k *the curvature of* C *at* p, *and* $\cos\theta = \langle n, N\rangle$, *where* n *is the normal vector to* C *and* N *is the normal vector to* S *at* p. *The number* $k_n = k\cos\theta$ *is then called the* normal curvature *of* C $\subset$ S *at* p.

In other words, k_n is the length of the projection of the vector kn over the normal to the surface at p, with a sign given by the orientation N of S at p (Fig. 3-8).



Projection of curvature to the surface.

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Sectional curvature.

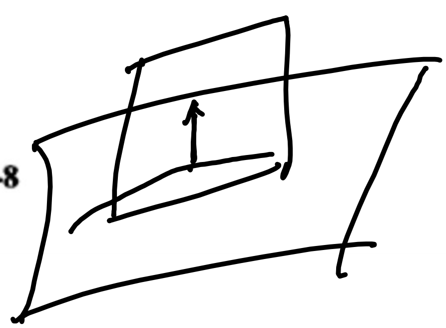

Figure 3-8

Relations :

To give an interpretation of the second fundamental form II_p, consider a regular curve $C \subset S$ parametrized by $\alpha(s)$, where s is the arc length of C, and with $\alpha(0) = p$. If we denote by $N(s)$ the restriction of the normal vector N to the curve $\alpha(s)$, we have $\langle N(s), \alpha'(s)\rangle = 0$. Hence,

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$$\langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle.$$

Therefore,

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle \\ &= \langle N, kn\rangle(p) = k_n(p). \end{aligned}$$

In other words, the value of the second fundamental form II_p for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v. In particular, we obtained the following result.

Let us come back to the linear map dN_p. The theorem of the appendix to Chap. 3 shows that for each $p \in S$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1e_1$, $dN_p(e_2) = -k_2e_2$. Moreover, k_1 and k_2 $(k_1 \geq k_2)$ are the maximum and minimum of the second fundamental form II_p restricted to the unit circle of $T_p(S)$; that is, they are the extreme values of the normal curvature at p.

~~DEFINITION 4.~~ Theorem *The maximum normal curvature* k_1 *and the minimum normal curvature* k_2 *are called the* principal curvatures *at* p*; the corresponding directions, that is, the directions given by the eigenvectors* e_1, e_2*, are called* principal directions *at* p.

Check: find max. & min. of $k(u,v)$.

DEFINITION 6. *Let* $p \in S$ *and let* dN_p*:* $T_p(S) \rightarrow T_p(S)$ *be the differential of the Gauss map. The determinant of* dN_p *is the* Gaussian curvature K *of* S *at* p*. The negative of half of the trace of* dN_p *is called the* mean curvature H *of* S *at* p.

In terms of the principal curvatures we can write

$$K = k_1 k_2, \qquad H = \frac{k_1 + k_2}{2}.$$

DEFINITION 7. *A point of a surface* S *is called*

1. Elliptic *if* $det(dN_p) > 0$.
2. Hyperbolic *if* $det(dN_p) < 0$.
3. Parabolic *if* $det(dN_p) = 0$, *with* $dN_p \neq 0$.
4. Planar *if* $dN_p = 0$.